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Periodic Boundary Value Problem for Differential Equations with Delay and Monotone Iterative Method

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1. INTRODUCTION

Existence of extremal solutions for a variety of nonlinear differential equations is studied by a combined approach of the method of lower and upper solutions and the monotone iterative technique [3]. It is natural to extend this useful method to delay differential equations. However, this extension creates serveral difficulties. For example, we need a contraction mapping theorem for nonlinear operators whose domain and range are different Banach spaces and the theory of corresponding differential inequalities poses considerable problems. In this paper, we discuss the existence of extremal solutions of differential equations with delay using periodic boundary conditions of a particular type.

2. MONOTONE METHOD

Consider the delay-differential equation

$$x' = f(t, x, x_t) \tag{2.1}$$

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subject to the boundary condition

$$x(0) = x(2\pi) \tag{2.2}$$

where $f \in C[I \times R \times \mathscr{C}, R]$, $I = [0, 2\pi]$, $\mathscr{C} = C[[-\tau, 0], R]$, $\tau > 0$ ($\mathscr{C}$ is the space of continuous functions $\phi: [-\tau, 0] \to R$ with the norm $|\phi|_0 = \max_{[-\tau,0]} |\phi(\theta)|$) and $x_t \in \mathscr{C}$ is defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leqslant \theta \leqslant 0$. Let $\alpha, \beta \in E$ be lower and upper solutions relative to (2.1), (2.2) with $\alpha(t) \leqslant \beta(t)$ on I, where $E = C[[-\tau, 2\pi], R] \cap C^1[[0, 2\pi], R]$. Assume that α, β satisfy the following hypothesis (H_0):

(H_0) $\alpha' \leqslant f(t, \alpha, \alpha_t) - v_\alpha$, $\beta' \geqslant f(t, \beta, \beta_t) + v_\beta$, $t \in I$, where for $M > 0$ and $\lambda = Me^{2M\pi}/e^{2M\pi} - 1$,

$$v_\alpha = \begin{cases} 0, & \text{if } \alpha(0) \leqslant \alpha(2\pi), \\ [\alpha(0) - \alpha(2\pi)]\lambda, & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$v_\beta = \begin{cases} 0, & \text{if } \beta(0) \geqslant \beta(2\pi), \\ [\beta(2\pi) - \beta(0)]\lambda, & \text{if } \beta(0) < \beta(2\pi). \end{cases}$$

In order to develop the monotone method for the periodic boundary value problem (PBVP for brevity) (2.1), (2.2), we require that f satisfies hypothesis (H_1):

$$(\mathrm{H}_1) \qquad f(t, u, \phi) - f(t, \bar{u}, \bar{\phi}) \geqslant -M(u - \bar{u}) + N \int_{t-\tau}^{t} [\phi(s) - \bar{\phi}(s)]\, ds,$$

whenever $\alpha(t) \leqslant \bar{u} \leqslant u \leqslant \beta(t)$, $\alpha_t(\theta) \leqslant \bar{\phi}(\theta) \leqslant \phi(\theta) \leqslant \beta_t(\theta)$, $t \in [0, 2\pi]$, $\theta \in [-\tau, 0]$ and $M, N > 0$ are such that $M > N(2\pi + \tau)$.

In the sequel, we need the following contraction mapping theorem for operators whose domain and range are different Banach spaces.

THEOREM 2.1. *Let F be a Banach space and $\hat{E} = C[[a, b], F]$. Let $S: \hat{E} \to F$ be an operator satisfying*

$$\|S\phi - S\psi\|_F \leqslant \alpha \|\phi - \psi\|_{\hat{E}}, \qquad 0 \leqslant \alpha < 1.$$

Then, for any given $\hat{c} \in [a, b]$, there exists a $\phi \in \hat{E}$ such that $S\phi = \phi(\hat{c})$.

For details cencerning such a fixed point theorem and its ramifications, see [1].

Let us now take $\hat{E}$ to be the Banach space characterized by $E_0 = \{x \in C[[-\tau, 2\pi], R]: x(\theta) \equiv x(0) \text{ on } [-\tau, 0]\}$ with $|x|_0 = \max_{[-\tau,2\pi]} |x(s)|$. We shall now prove a comparison lemma which plays a crucial role in our results.

LEMMA 2.1. *Assume that* $m \in E \cap E_0$ *and satisfies the differential inequality*

$$m'(t) \leqslant -Mm(t) + N \int_{t-\tau}^{t} m(s)\, ds - v_m, \qquad t \in [0, 2\pi] \tag{2.3}$$

where

$$v_m = \begin{cases} 0, & \text{if } m(0) \leqslant m(2\pi), \\ [m(0) - m(2\pi)]\lambda, & \text{if } m(0) > m(2\pi), \end{cases}$$

where $M, N > 0$ *such that* $M > N(2\pi + \tau)$ *and* $\lambda = Me^{2M\pi}/(e^{2M\pi} - 1)$. *Then* $m(t) \leqslant 0$ *on* $[0, 2\pi]$.

Proof. If the conclusion is not true then there must exist a $t_0 \in [0, 2\pi]$ and an $\varepsilon > 0$ such that

$$m(t_0) = \varepsilon \qquad \text{and} \qquad m(t) \leqslant \varepsilon,\ t \in [0, 2\pi]. \tag{2.4}$$

Let $t_0 \in (0, 2\pi]$. Then, $m'(t_0) \geqslant 0$ and consequently,

$$0 \leqslant m'(t_0) \leqslant -Mm(t_0) + N \int_{t_0-\tau}^{t_0} m(s)\, ds - v_m. \tag{2.5}$$

If $t_0 - \tau \geqslant 0$, using (2.4) and (2.5) we arrive at the contradiction

$$0 \leqslant m'(t_0) \leqslant -M\varepsilon + N\tau\varepsilon - v_m < 0,$$

since $M > N(2\pi + \tau)$ implies $M > N\tau$. If $t_0 - \tau < 0$, then (2.4), (2.5) and the fact that $m(\theta) \equiv m(0)$ on $[-\tau, 0]$ yield

$$\begin{aligned} 0 \leqslant m'(t_0) &\leqslant -M\varepsilon + N\left(\int_{t_0-\tau}^{0} m(s)\, ds + \int_{0}^{t_0} m(s)\, ds\right) - v_m \\ &\leqslant -M\varepsilon + Nm(0)(\tau - t_0) + Nt_0\varepsilon - v_m \\ &\leqslant [-M + N(\tau + 2\pi)]\varepsilon - v_m < 0, \end{aligned}$$

which is a contradiction.

On the other hand if $t_0 = 0$ and $\varepsilon = m(0) \leqslant m(2\pi)$, we get

$$\begin{aligned} 0 \leqslant m'(2\pi) &\leqslant -Mm(2\pi) + N \int_{2\pi-\tau}^{2\pi} m(s)\, ds \\ &\leqslant -M\varepsilon + N\tau\varepsilon < 0. \end{aligned}$$

However, if $t_0 = 0$ and $\varepsilon = m(0) > m(2\pi)$, we can multiply both sides of (2.3) by e^{Mt} and integrate on $[0, 2\pi]$ to obtain

$$m(2\pi)\, e^{2M\pi} - m(0) \leqslant N \int_0^{2\pi} e^{Ms} \left[\int_{s-\tau}^{0} m(0)\, d\xi + \int_0^s m(\xi)\, d\xi \right] ds$$
$$- [m(0) - m(2\pi)]\, e^{2m\pi},$$

which implies $m(0)[e^{2M\pi} - 1](1 - (N\tau/M)) < 0$. This contradicts $m(0) = \varepsilon$ and hence the proof of the lemma is complete.

Before we state our main result on monotone iterates, let us prove the following lemmas.

LEMMA 2.2. *Let* $[\alpha, \beta]$ *denote the sector* $\{\eta \in E \cap E_0 : \alpha \leqslant \eta \leqslant \beta\}$. *For any* $\eta \in [\alpha, \beta]$, *the* PBVP

$$\begin{aligned} u' &= f(t, \eta, \eta_t) - M(u - \eta) + N \int_{t-\tau}^{t} [u(s) - \eta(s)]\, ds, \qquad t \in I \\ u(0) &= u(2\pi) \end{aligned} \tag{2.6}$$

has a unique solution in E_0 *provided* $N(2\pi + \tau) < M$.

Proof. The PBVP (2.6) can be written as

$$u' + Mu = \sigma(t) + N \int_{t-\tau}^{t} u(s)\, ds, \qquad t \in I,\ u(0) = u(2\pi),$$

where $\sigma(t) = f(t, \eta, \eta_t) + M\eta - N \int_{t-\tau}^{t} \eta(s)\, ds$. Using the method of variation of parameters and the boundary condition, we get

$$u(0) = u(2\pi) = \frac{1}{e^{2M\pi} - 1} \left(\int_0^{2\pi} \left[\sigma(s) + N \int_{s-\tau}^{s} u(\xi)\, d\xi \right] e^{Ms}\, ds \right.$$
$$\overset{\text{def}}{\equiv} Su.$$

Clearly the operator S is defined from E_0 into R. Let us note that

$$|Su - Sv| \leqslant \frac{N}{e^{2M\pi} - 1} \left(\int_0^{2\pi} \left[\int_{-\tau}^{2\pi} |u - v|_0\, d\xi \right] e^{Ms} \right) ds$$
$$= \frac{N(2\pi + \tau)}{M} |u - v|_0,$$

which shows that S is a contradiction mapping whenever $N(2\pi + \tau) < M$. Hence, by Theorem 2.1, the operator S has a fixed point in E_0, i.e., there exists a solution u of PBVP (2.6) such that $u(\theta) \equiv u(0) = u(2\pi)$, $\theta \in [-\tau, 0]$.

The uniqueness of solutions of PBVP (2.6) follows from Lemma 2.1. In fact, if $u_1, u_2 \in E_0$ are two distinct solutions of (2.6), setting $m(t) = u_1(t) - u_2(t)$, $t \in [-\tau, 2\pi]$, we see that

$$m'(t) = -Mm(t) + N \int_{t-\tau}^{t} m(s)\, ds, \qquad t \in I,$$

and $m(\theta) \equiv m(0) = m(2\pi)$, $\theta \in [-\tau, 0]$. Hence, by Lemma 2.1, $m(t) \leqslant 0$ which implies $u_1(t) \leqslant u_2(t)$, $t \in I$. On the other hand, setting $m(t) = u_2(t) - u_1(t)$ and following the same argument, we can show that $u_2(t) \leqslant u_1(t)$. Thus $u_1(t) \equiv u_2(t)$ on $[-\tau, 2\pi]$.

LEMMA 2.3. *Let* $\alpha, \beta \in E \cap E_0$ *with* $\alpha(t) \leqslant \beta(t)$ *on* I. *Suppose that hypotheses* (H_0) *and* (H_1) *hold. Then, the mapping* A *defined by* $A\eta = u$, *where for any* $\eta \in [\alpha, \beta]$, u *is the unique solution of* PBVP (2.6), *has the following properties*:

(i) $\alpha \leqslant A\alpha$, $\beta \geqslant A\beta$;

(ii) A *is monotone nondecreasing on the sector* $[\alpha, \beta]$, *i.e., for any*

$$\eta_1, \eta_2 \in [\alpha, \beta], \eta_1 \leqslant \eta_2 \qquad \textit{implies} \quad A\eta_1 \leqslant A\eta_2.$$

Proof. To prove (i), set $m = \alpha - \alpha_1$, where $\alpha_1 = A\alpha$. We then have, using (H_0) and (2.6),

$$m'(t) \leqslant -Mm(t) + N \int_{t-\tau}^{t} m(s)\, ds, \qquad t \in I$$

with $m(\theta) \equiv m(0) \leqslant m(2\pi)$ whenever $\alpha(0) \leqslant \alpha(2\pi)$, or

$$m'(t) \leqslant -Mm(t) + N \int_{t-\tau}^{t} m(s)\, ds - [m(0) - m(2\pi)]\lambda, \qquad t \in I,$$

with $m(\theta) \equiv m(0) > m(2\pi)$ whenever $\alpha(0) > \alpha(2\pi)$.

Hence, in either case, by Lemma 2.1, we get $m(t) \leqslant 0$, proving $\alpha \leqslant A\alpha$. Similar arguments for $m = \beta_1 - \beta$ show that $\beta \geqslant A\beta$. To prove (ii), let $A\eta_1 = u_1$ and $A\eta_2 = u_2$ where $\eta_1 \leqslant \eta_2$ on $[-\tau, 2\pi]$ and $\eta_1, \eta_2 \in [\alpha, \beta]$. Set $m = u_1 - u_2$. Then, it follows, using (H_1) and (2.6), that

$$m'(t) \leqslant -Mm(t) + N \int_{t-\tau}^{t} m(s)\, ds, \qquad t \in I$$

and $m(\theta) \equiv m(0) = m(2\pi)$, $\theta \in [-\tau, 0]$, which in view of Lemma 2.1 implies that $u_1(t) \leqslant u_2(t)$ on $[-\tau, 2\pi]$.

Now, we can state our main result.

THEOREM 2.2. *Assume that* (H_0) *and* (H_1) *hold relative to lower and upper solutions* $\alpha, \beta \in E \cap E_0$ *with* $\alpha(t) \leqslant \beta(t)$ *on* I. *Then, there exists monotone sequences* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *with* $\alpha_0 \equiv \alpha$, $\beta_0 \equiv \beta$ *such that* $\lim_{n \to \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \to \infty} \beta_n(t) = r(t)$ *uniformly on* $[0, 2\pi]$ *and* ρ, r *are minimal and maximal solutions of* PBVP (2.1), (2.2) *respectively. The solutions* $\rho, r \in E_0$ *in the sense that* $\rho(\theta) \equiv \rho(0) = \rho(2\pi)$, $r(\theta) \equiv r(0) = r(2\pi)$, $\theta \in [-\tau, 0]$ *and* $\rho(t) \leqslant x(t) \leqslant r(t)$, $t \in I$, *where* x *is any solution of* (2.1), (2.2).

Proof. By Lemma 2.2, we obtain that for any $\eta \in [\alpha, \beta]$, the problem (2.6) has a unique solution $u \in E_0$. By Lemma 2.3, we see that the mapping $A\eta = u$ generates the monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ where $A\alpha_n = \alpha_{n+1}$, $A\beta_n = \beta_{n+1}$ such that on $[-\tau, 2\pi]$,

$$\alpha \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant \cdots \leqslant \beta_n \leqslant \cdots \leqslant \beta_2 \leqslant \beta_1 \leqslant \beta$$

and each $\alpha_n, \beta_n \in E_0$, $n = 1, 2, 3, \ldots$, satisfies the PBVP

$$\begin{aligned} \alpha_n'(t) &= f(t, \alpha_{n-1}, \alpha_{n-1t}) - M(\alpha_n - \alpha_{n-1}) \\ &\quad + N \int_{t-\tau}^{t} [\alpha_n(s) - \alpha_{n-1}(s)]\, ds, \qquad t \in I, \\ \alpha_n(0) &= \alpha_n(2\pi), \\ \beta_n'(t) &= f(t, \beta_{n-1}, \beta_{n-1t}) - M(\beta_n - \beta_{n-1}) \\ &\quad + N \int_{t-\tau}^{t} [\beta_n(s) - \beta_{n-1}(s)]\, ds, \qquad t \in I, \\ \beta_n(0) &= \beta_n(2\pi), \end{aligned}$$

respectively. It now follows by using standard arguments [3], that $\lim_{n \to \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \to \infty} \beta_n(t) = r(t)$ uniformly on $[-\tau, 2\pi]$ and that ρ, r are extremal solutions of PBVP (2.1), (2.2). The proof of the theorem is complete.

Remarks. (a) Perhaps a natural PBVP for the delay-differential equation (2.1) is to require the boundary condition to be

$$x_0(\theta) = x_{2\pi}(\theta), \qquad -\tau \leqslant \theta \leqslant 0,$$

so that if f is 2π-periodic in t, then we get a periodic solution of (2.1). The monotone iterative technique coupled with the methods of lower and upper solutions does not seem to be a suitable approach for this kind of PBVP even with more restrictive conditions. We note that in order to show the existence of a solution for the linear problem (2.6), we could relax the space E_0 to be $C[[-\tau, 2\pi], R]$ itself. Our choice of E_0 is dictated by the difficulties in the proof of the comparison result (Lemma 2.1) which is crucial for our method. The fact that there exists a solution u of (2.6) does not guarantee that $\alpha \leqslant u \leqslant \beta$ even at a single point of $[-\tau, 2\pi]$, if we take $C[[-\tau, 2\pi], R]$. But proving this inequality is the central idea in generating the monotone iterates that lie in the sector and consequently we have the necessity of restricting ourselves to the choice of E_0 and the type of PBVP that is considered in our foregoing discussion. This choice of E_0 restricts the class of initial functions to constant functions on $[-\tau, 0]$. Furthermore, our formulation of PBVP does not allow to obtain periodic solutions when f is periodic in t. However we have attempted to obtain a partial answer to the question raised in [2].

(b) In the case of lower and upper solutions satisfying (H_0) with $\beta \leqslant \alpha$, Theorem 2.2 holds good with the hypothesis (H_1) modified as

$$(H_1^*) \qquad f(t, u, \phi) - f(t, \bar{u}, \bar{\phi}) \leqslant M(u - \bar{u}) - N \int_{t-\tau}^{t} [\phi(s) - \bar{\phi}(s)]\, ds$$

whenever $\beta(t) \leqslant \bar{u} \leqslant u \leqslant \alpha(t)$, $\beta_t(\theta) \leqslant \bar{\phi}(\theta) \leqslant \phi(\theta) \leqslant \alpha_t(\theta)$, $t \in [0, 2\pi]$ and $\theta \in [-\tau, 0]$.

It is also clear that Theorem 2.2 is valid when the boundary conditions that α and β must satisfy is any one of the following four relations: (i) $\alpha(0) \leqslant \alpha(2\pi)$, $\beta(0) \geqslant \beta(2\pi)$; (ii) $\alpha(0) > \alpha(2\pi)$, $\beta(0) \geqslant \beta(2\pi)$; (iii) $\alpha(0) \geqslant \alpha(2\pi)$, $\beta(0) < \beta(2\pi)$ and (iv) $\alpha(0) > \alpha(2\pi)$, $\beta(0) < \beta(2\pi)$.

REFERENCES

1. S. R. BERNFELD, V. LAKSHMIKANTHAM, AND Y. M., REDDY, Fixed point theorems for operators with PPF dependence in Banach spaces, *Appl. Anal.* **6** (1977), 271–280.
2. V. LAKSHMIKANTHAM, Periodic boundary value problems for first and second-order differential equations, to appear.
3. V. LAKSHMIKANTHAM, G. S. LADDE, AND A. S. VATSALA, "Monotone Iterative Techniques for Nonlinear Differential Equations," Pitman, London, in press.